Question 1 (Markov Chain):

1. P is the transition matrix for some Markov Chain (MC), so is the transition probability from state I to j. Thus, for all i,j.

Each row in P sums up to 1 (i.e. P is row stochastic):

The first equality holds since we consider a time-homogeneous MC as defined in class:

1. We show that .

P is row stochastic, then for each row . It means that is one of the eigenvalues of P and the corresponding eigenvector is . Since the right eigenvalues and left eigenvalues are the same for square matrices, 1 is also a left eigenvalue of P.

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1. Let be a P eigenvalue with the corresponding eigenvector. We show that .

Suppose by contradiction that . Let’s denote .

Since any holds the equation, then we assume .

But in the second hand, P rows sum to 1 and each element in is a convex combination of . Thus, no entry in can be larger than . Contradiction.

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Question 3

1. is the probability that the tth candidate has the highest score while , meaning that the current candidate is not the best. So obviously, the tth candidate has not the highest score:

For s=1, we have seen so far t-1 candidates, and we are interested in the probability that the tth candidate has highest score. Of course if the tth candidate has highest score, it has particularly the highest among first t candidates and that’s the information we are given (as interviewers)

From uniform sampling we have:

1. Now we are interested in . It is the probability that the t+1th candidate is the best one given that we already interviewed t candidates. Each candidate is uniformly sampled, thus .

And .

1. To compute , we need to consider two options at time t and state s: hire tth candidate or continue interviewing.

The first option to pick the tth candidate after interviewing t candidates and we are in state s, that it .

The second option is to continue searching and act according to In the next step, we may be in two different states. So, if we continue searching V holds the following.

Of course, we’ll act greedily at each time step and take the maximum between the two options:

since at time N if we have not chosen any candidate and the last one is the best, then the probability to choose the best is 1.

from similar considerations, at last time step, if the last candidate is not the best, then the probability to choose the best is 0.

1. Since :

The code of the plot appears at the end. The plot if V values for N=10 is:

Chart, line chart

Description automatically generated

1. We observe that the values of Vt(1) and Vt(0) are the same until . It means that there is no value to choose a candidate before interviewing at least 4 candidates. Then meaning that the best option was to choose the tth candidate if he’s better than the previous ones.

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